Variance Control in Weak Value Measurement Pointers

A. D. Parks and J. E. Gray

Electromagnetic and Sensor Systems Department,

Naval Surface Warfare Center, Dahlgren, VA 22448, USA

Abstract

The variance of an arbitrary pointer observable is considered for the general case that a complex weak value is measured using a complex valued pointer state. For the typical cases where the pointer observable is either its position or momentum, the associated expressions for the pointer's variance after the measurement contain a term proportional to the product of the weak value's imaginary part with the rate of change of the third central moment of position relative to the initial pointer state just prior to the time of the measurement interaction when position is the observable - or with the initial pointer state's third central moment of momentum when momentum is the observable. These terms provide a means for controlling pointer position and momentum variance and identify control conditions which - when satisfied - can yield variances that are smaller after the measurement than they were before the measurement. Measurement sensitivities which are useful for estimating weak value measurement accuracies are also briefly discussed.

I. INTRODUCTION

The weak value A_w of a quantum mechanical observable A was introduced by Aharonov *et al* [1–3] a quarter century ago. This quantity is the statistical result of a standard measurement procedure performed upon a pre- and post-selected (PPS) ensemble of quantum systems when the interaction between the measurement apparatus and each system is sufficiently weak, i.e. when it is a weak measurement. Unlike a standard strong measurement of A which significantly disturbs the measured system (i.e. it "collapses" the wavefunction), a weak measurement of A for a PPS system does not appreciably disturb the quantum system and yields A_w as the observable's measured value. The peculiar nature of the virtually undisturbed quantum reality that exists between the boundaries defined by the PPS states is revealed in the eccentric characteristics of A_w, namely that A_w can be complex valued and that the real part $\operatorname{Re} A_w$ of A_w can lie far outside the eigenvalue spectral limits of $\widehat{A}$. While the interpretation of weak values remains somewhat controversial, experiments have verified several of the interesting unusual properties predicted by weak value theory [4–10].

The pointer of a measurement apparatus is fundamental to the theory of quantum measurement because the values of measured observables are determined from the pointer's properties (e.g. from the pointer's mean position). Understanding these properties has become more important in recent years - in large part due to the increased interest in the theory of weak measurements and weak value theory. The properties of pointers associated with weak value measurements have been studied - for example - by Johansen [11], Aharonov and Botero [12], Jozsa [13], Di Lorenzo and Egues [14], and Cho *et al* [15].

The purpose of this paper is to extend Jozsa's work [13] to obtain the general expression for the variance associated with an arbitrary pointer observable when a complex valued pointer state is used to measure a complex weak value A_w. For the typical cases where position or momentum are the pointer observables, the associated expressions each contain a variance control term. This term is proportional to the product of the imaginary part $\operatorname{Im} A_w$ of A_w with the rate of change of the third central moment of position relative to the initial pointer state just prior to measurement when the observable is position - or with the initial pointer state's third central moment of momentum when momentum is the observable. Control conditions associated with these terms are identified which - if satisfied - can yield pointer position and momentum variances after a measurement that are

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smaller than they were prior to the measurement. These results are used to briefly discuss sensitivities associated with weak value measurements.

II.   WEAK MEASUREMENTS AND WEAK VALUES

For the reader's convenience, this section provides a brief review of weak measurement and weak value theory. For additional details the reader is invited to consult references [1–3, 16].

Weak measurements arise in the von Neumann description of a quantum measurement at time t_0 of a time-independent observable A that describes a quantum system in an initial fixed pre-selected state $|\psi_i\rangle = \sum_J c_j |a_j\rangle$ at t_0, where the set J indexes the eigenstates $|a_j\rangle$ of $\widehat{A}$. In this description, the Hamiltonian for the interaction between the measurement apparatus and the quantum system is

$$\widehat{H} = \gamma(t)\widehat{A}\widehat{p}.$$

Here $\gamma(t) = \gamma\delta(t - t_0)$ defines the strength of the measurement's impulsive coupling interaction at t_0 and $\widehat{p}$ is the momentum operator for the pointer of the measurement apparatus which is in the initial normalized state $|\phi\rangle$. Let $\widehat{q}$ be the pointer's position operator that is conjugate to $\widehat{p}$.

Prior to the measurement the pre-selected system and the pointer are in the tensor product state $|\psi_i\rangle |\phi\rangle$. Immediately following the interaction the combined system is in the state

$$|\Phi\rangle = e^{-\frac{i}{\hbar}\int \widehat{H}dt} |\psi_i\rangle |\phi\rangle = \sum_J c_j e^{-\frac{i}{\hbar}\gamma a_j \widehat{p}} |a_j\rangle |\phi\rangle,$$

where use has been made of the fact that $\int \widehat{H}dt = \gamma\widehat{A}\widehat{p}$. The exponential factor in this equation is the translation operator $\widehat{S}(\gamma a_j)$ for $|\phi\rangle$ in its q-representation. It is defined by the action $\langle q| \widehat{S}(\gamma a_j) |\phi\rangle$ which translates the pointer's wavefunction over a distance γa_j parallel to the q-axis. The q-representation of the combined system and pointer state is

$$\langle q| \Phi\rangle = \sum_J c_j \langle q| \widehat{S}(\gamma a_j) |\phi\rangle |a_j\rangle.$$

When the measurement interaction is strong, the quantum system is appreciably disturbed and its state "collapses" with probability $|c_n|^2$ to an eigenstate $|a_n\rangle$ leaving the

pointer in the state $\langle q| \widehat{S}(\gamma a_n) |\phi\rangle$. Strong measurements of an ensemble of identically pre-pared systems yield $\gamma \langle A \rangle \equiv \gamma \langle \psi_i| \widehat{A} |\psi_i\rangle$ as the centroid of the associated pointer probability distribution with $\langle A \rangle$ as the measured value of $\widehat{A}$.

A *weak measurement* of $\widehat{A}$ occurs when the interaction strength γ is sufficiently small so that the system is essentially undisturbed and the uncertainty Δq is much larger than $\widehat{A}$'s eigenvalue separation. In this case, the pointer distribution is the superposition of broad overlapping $\left| \langle q| \widehat{S}(\gamma a_j) |\phi\rangle \right|^2$ terms. Although a single measurement provides little informa-tion about $\widehat{A}$, many repetitions allow the centroid of the distribution to be determined to any desired accuracy.

If a system is post-selected after a weak measurement is performed, then the resulting pointer state is

$$|\Psi\rangle \equiv \langle \psi_f| \Phi\rangle = \sum_J c_j'^* c_j \widehat{S}(\gamma a_j) |\phi\rangle,$$

where $|\psi_f\rangle = \sum_J c_j' |a_j\rangle$, $\langle \psi_f| \psi_i\rangle \neq 0$, is the post-selected state at t_0. Since

$$\widehat{S}(\gamma a_j) = \sum_{m=0}^{\infty} \frac{(-i\gamma a_j \widehat{p}/\hbar)^m}{m!},$$

then

$$|\Psi\rangle = \sum_J c_j'^* c_j \left\{ 1 - \frac{i}{\hbar} \gamma A_w \widehat{p} + \sum_{m=2}^{\infty} \frac{(-i\gamma \widehat{p}/\hbar)^m}{m!} (A^m)_w \right\} |\phi\rangle \approx \left\{ \sum_J c_j'^* c_j \right\} e^{-\frac{i}{\hbar} \gamma A_w \widehat{p}} |\phi\rangle$$

in which case

$$|\Psi\rangle \approx \langle \psi_f| \psi_i\rangle \widehat{S}(\gamma A_w) |\phi\rangle.$$

Here

$$(A^m)_w = \frac{\sum_J c_j'^* c_j a_j^m}{\sum_J c_j'^* c_j} = \frac{\langle \psi_f| \widehat{A}^m |\psi_i\rangle}{\langle \psi_f| \psi_i\rangle},$$

with the *weak value* A_w of $\widehat{A}$ defined by

$$A_w \equiv \left(A^1 \right)_w = \frac{\langle \psi_f| \widehat{A} |\psi_i\rangle}{\langle \psi_f| \psi_i\rangle}. \tag{1}$$

From this expression it is obvious that A_w is - in general - a complex valued quantity that can be calculated directly from theory and that when the PPS states are nearly orthogonal $\mathrm{Re}\, A_w$ can lie far outside $\widehat{A}$'s eigenvalue spectral limits.

For the general case where both A_w and $\phi(q)$ are complex valued, the mean pointer position and momentum after a measurement are given by [13]

$$\langle\Psi|\,\widehat{q}\,|\Psi\rangle = \langle\phi|\,\widehat{q}\,|\phi\rangle + \gamma\operatorname{Re}A_w + \left(\frac{\gamma}{\hbar}\right)\operatorname{Im}A_w\left(m\frac{d\Delta_\phi^2 q}{dt}\right) \tag{2}$$

and

$$\langle\Psi|\,\widehat{p}\,|\Psi\rangle = \langle\phi|\,\widehat{p}\,|\phi\rangle + 2\left(\frac{\gamma}{\hbar}\right)\operatorname{Im}A_w\left(\Delta_\phi^2 p\right), \tag{3}$$

respectively. Here m is the mass of the pointer, $\Delta_\phi^2 p$ is the pointer's initial momentum variance, and the time derivative of $\Delta_\phi^2 q$ is the rate of change of the initial pointer position variance just prior to t_0.

III. POINTER VARIANCE

The mean value of an arbitrary pointer observable M after a measurement of A_w is [13]

$$\langle\Psi|\,\widehat{M}\,|\Psi\rangle = \langle\phi|\,\widehat{M}\,|\phi\rangle - i\left(\frac{\gamma}{\hbar}\right)\operatorname{Re}A_w\langle\phi|\left[\widehat{M},\widehat{p}\right]|\phi\rangle + \tag{4}$$
$$\left(\frac{\gamma}{\hbar}\right)\operatorname{Im}A_w\left(\langle\phi|\left\{\widehat{M},\widehat{p}\right\}|\phi\rangle - 2\langle\phi|\,\widehat{M}\,|\phi\rangle\langle\phi|\,\widehat{p}\,|\phi\rangle\right),$$

where $\left\{\widehat{M},\widehat{p}\right\} = \widehat{M}\widehat{p} + \widehat{p}\widehat{M}$. Note that eq.(4) reduces to eq.(3) when $\widehat{M} = \widehat{p}$ and that it is also in complete agreement with eq.(2) when $\widehat{M} = \widehat{q}$ since $[\widehat{q},\widehat{p}] = i\hbar$ and the equations of motion for $\langle\phi|\,\widehat{q}\,|\phi\rangle$ and $\langle\phi|\,\widehat{q}^2\,|\phi\rangle$ yield

$$\langle\phi|\left\{\widehat{q},\widehat{p}\right\}|\phi\rangle = m\frac{d\langle\phi|\,\widehat{q}^2\,|\phi\rangle}{dt} \tag{5}$$

and

$$2\langle\phi|\,\widehat{q}\,|\phi\rangle\langle\phi|\,\widehat{p}\,|\phi\rangle = m\frac{d\langle\phi|\,\widehat{q}\,|\phi\rangle^2}{dt}. \tag{6}$$

Here the time derivatives are rates of change of the corresponding quantities just prior to the interaction time t_0.

The pointer variance for M is easily determined from eq.(4) by subtracting its square from the expression obtained from eq.(4) when $\widehat{M}$ is replaced by $\widehat{M}^2$. Retaining terms through first order in $\left(\frac{\gamma}{\hbar}\right)$ yields the following result:

$$\Delta_\Psi^2 M = \Delta_\phi^2 M - i\left(\frac{\gamma}{\hbar}\right)\operatorname{Re}A_w\mathcal{F}\left(\widehat{M}\right) + \left(\frac{\gamma}{\hbar}\right)\operatorname{Im}A_w\mathcal{G}\left(\widehat{M}\right). \tag{7}$$

Here $\Delta_\phi^2 M$ and $\Delta_\Psi^2 M$ are the initial and final variances, respectively,

$$\mathcal{F}\left(\widehat{M}\right) \equiv \langle\phi|\left[\widehat{M}^2,\widehat{p}\right]|\phi\rangle - 2\langle\phi|\widehat{M}|\phi\rangle\langle\phi|\left[\widehat{M},\widehat{p}\right]|\phi\rangle,$$

and

$$\mathcal{G}\left(\widehat{M}\right) \equiv \langle\phi|\left\{\widehat{M}^2,\widehat{p}\right\}|\phi\rangle - 2\langle\phi|\widehat{M}|\phi\rangle\langle\phi|\left\{\widehat{M},\widehat{p}\right\}|\phi\rangle - 2\langle\phi|\widehat{p}|\phi\rangle\left(\Delta_\phi^2 M - \langle\phi|\widehat{M}|\phi\rangle^2\right). \tag{8}$$

As anticipated from eq.(4), eq.(7) clearly shows that for such a measurement the pointer variance associated with an arbitrary pointer observable is also generally effected by both the real and imaginary parts of the weak value.

However, for the typical cases of interest where $\widehat{M} = \widehat{q}$ or $\widehat{M} = \widehat{p}$, the pointer's variance is independent of $\mathrm{Re}\, A_w$ because

$$\mathcal{F}\left(\widehat{q}\right) = 0 = \mathcal{F}\left(\widehat{p}\right).$$

Here use has been made of the facts that $[\widehat{q},\widehat{p}] = i\hbar$ and $[\widehat{q}^2,\widehat{p}] = 2i\hbar\widehat{q}$. Consequently, for these cases eq.(7) can be written as

$$\Delta_\Psi^2 M = \Delta_\phi^2 M + \left(\frac{\gamma}{\hbar}\right)\mathrm{Im}\, A_w \mathcal{G}\left(\widehat{M}\right),\ M = q, p. \tag{9}$$

Now consider $\mathcal{G}\left(\widehat{M}\right)$ in more detail. When $\widehat{M} = \widehat{q}$, then eq.(8) becomes

$$\mathcal{G}\left(\widehat{q}\right) = \langle\phi|\left\{\widehat{q}^2,\widehat{p}\right\}|\phi\rangle - 2\langle\phi|\widehat{q}|\phi\rangle\langle\phi|\left\{\widehat{q},\widehat{p}\right\}|\phi\rangle - 2\langle\phi|\widehat{p}|\phi\rangle\left(\Delta_\phi^2 q - \langle\phi|\widehat{q}|\phi\rangle^2\right). \tag{10}$$

From the equation of motion for $\langle\phi|\widehat{q}^3|\phi\rangle$ it is found that

$$\frac{d\langle\phi|\widehat{q}^3|\phi\rangle}{dt} = -\frac{i}{\hbar}\langle\phi|\left[\widehat{q}^3,\widehat{H}\right]|\phi\rangle = -\frac{i}{2m\hbar}\langle\phi|\left[\widehat{q}^3,\widehat{p}^2\right]|\phi\rangle = \frac{3}{2m}\langle\phi|\left\{\widehat{q}^2,\widehat{p}\right\}|\phi\rangle,$$

where $\widehat{H} = \frac{\widehat{p}^2}{2m} + V(\widehat{q})$ is the pointer's Hamiltonian operator. Applying this result - along with eqs.(5) and (6) - to eq.(10) yields

$$\mathcal{G}\left(\widehat{q}\right) = \frac{2m}{3}\frac{dq_3}{dt},$$

so that eq.(9) can be compactly written as

$$\Delta_\Psi^2 q = \Delta_\phi^2 q + \frac{2\gamma m}{3\hbar}\mathrm{Im}\, A_w\left(\frac{dq_3}{dt}\right).$$

Here $q_3 \equiv \langle \phi | (\widehat{q} - \langle \phi | \widehat{q} | \phi \rangle)^3 | \phi \rangle$ is the third central moment of $\widehat{q}$ relative to the initial pointer state and its time derivative is the rate of change of q_3 just prior to t_0.

When $\widehat{M} = \widehat{p}$, then eq.(8) becomes

$$\mathcal{G}\left(\widehat{p}\right) = 2\left[\langle \phi | \widehat{p}^3 | \phi \rangle - 3 \langle \phi | \widehat{p} | \phi \rangle \langle \phi | \widehat{p}^2 | \phi \rangle + 2 \langle \phi | \widehat{p} | \phi \rangle^3\right] = 2p_3,$$

where $p_3 \equiv \langle \phi | (\widehat{p} - \langle \phi | \widehat{p} | \phi \rangle)^3 | \phi \rangle$ is the third central moment of $\widehat{p}$ relative to the pointer's initial state, and eq.(9) assumes the form

$$\Delta_\Psi^2 p = \Delta_\phi^2 p + 2 \left(\frac{\gamma}{\hbar}\right) \operatorname{Im} A_w (p_3).$$

The quantities q_3 and p_3 provide measures of the skewness of the pointer position and momentum probability distribution profiles. If the pointer position profile's skewness is fixed, then $\frac{dq_3}{dt} = 0$ and $\Delta_\Psi^2 q = \Delta_\phi^2 q$. Otherwise, $\Delta_\Psi^2 q$ can be manipulated through the judicious selection of the control term $\operatorname{Im} A_w \left(\frac{dq_3}{dt}\right)$. In particular, observe that $0 < \Delta_\Psi^2 q \leq \Delta_\phi^2 q$ when this control term satisfies the inequality

$$-\left(\frac{3\hbar}{2\gamma m}\right) \Delta_\phi^2 q < \operatorname{Im} A_w \left(\frac{dq_3}{dt}\right) \leq 0. \tag{11}$$

Similarly, the control term $\operatorname{Im} A_w (p_3)$ can be used to manipulate $\Delta_\Psi^2 p$ when it satisfies the inequality

$$-\left(\frac{\hbar}{2\gamma}\right) \Delta_\phi^2 p < \operatorname{Im} A_w (p_3) \leq 0. \tag{12}$$

Thus, when measuring complex weak values the final pointer position (momentum) variance can be made smaller than its initial value by choosing $\operatorname{Im} A_w$ or $\frac{dq_3}{dt}$ (p_3) so that condition (11) ((12)) is satisfied.

IV. CLOSING REMARKS

Because of the growing interest in the practical application of weak values, estimating their measurement sensitivities has also become important from both the experimental and device engineering perspectives. Applying the calculus of error propagation to the above results defines the measurement sensitivities $\delta_q \operatorname{Re} A_w$ and $\delta_q \operatorname{Im} A_w$ for determining $\operatorname{Re} A_w$ and $\operatorname{Im} A_w$ from the mean pointer position. These sensitivities are the positive square roots

of the following expressions:

$$\delta_q^2 \operatorname{Re} A_w \equiv \frac{\Delta_\Psi^2 q}{\left| \frac{\partial \langle \Psi | \hat{q} | \Psi \rangle}{\partial \operatorname{Re} A_w} \right|^2} = \frac{\Delta_\phi^2 q}{\gamma^2} + \frac{2m}{3\gamma\hbar} \operatorname{Im} A_w \left(\frac{dq_3}{dt} \right)$$

(13)

(this quantity is obviously undefined when A_w is purely imaginary) and

$$\delta_q^2 \operatorname{Im} A_w \equiv \frac{\Delta_\Psi^2 q}{\left| \frac{\partial \langle \Psi | \hat{q} | \Psi \rangle}{\partial \operatorname{Im} A_w} \right|^2} = \left(\frac{\hbar}{\gamma m} \right)^2 \left(\frac{\Delta_\phi^2 q}{\left| \frac{d\Delta_\phi^2 q}{dt} \right|^2} \right) + \frac{2}{3} \left(\frac{\hbar}{\gamma m} \right) \operatorname{Im} A_w \left(\frac{\frac{dq_3}{dt}}{\left| \frac{d\Delta_\phi^2 q}{dt} \right|^2} \right), \frac{d\Delta_\phi^2 q}{dt} \neq 0$$

(14)

(this quantity is obviously undefined when A_w is real valued or when $\frac{d\Delta_\phi^2 q}{dt} = 0$ - in which case the mean position does not depend upon $\operatorname{Im} A_w$). It is clear from eqs.(13) and (14) that: (i) these measurement sensitivities depend upon the variance control term $\operatorname{Im} A_w \left(\frac{dq_3}{dt} \right)$ and that this dependence vanishes when q_3 is fixed (or if A_w is real valued); (ii) these measurement accuracies decrease (increase) as the measurement gets weaker (stronger) - i.e. as γ gets smaller (larger); (iii) in principle - the accuracies associated with measuring both $\operatorname{Re} A_w$ and $\operatorname{Im} A_w$ can be arbitrarily increased (for a fixed $\gamma > 0$ and m) by invoking condition (11) and choosing $\operatorname{Im} A_w \left(\frac{dq_3}{dt} \right) = - \left(\frac{3\hbar}{2\gamma m} \right) \Delta_\phi^2 q + \epsilon$, where ϵ is a small positive real number; and (iv) surprisingly, the measurement accuracy for $\operatorname{Re} A_w$ decreases with increasing pointer mass, whereas that for $\operatorname{Im} A_w$ increases.

The sensitivity $\delta_p \operatorname{Im} A_w$ for determining $\operatorname{Im} A_w$ from the mean pointer momentum is the positive square root of

$$\delta_p^2 \operatorname{Im} A_w \equiv \frac{\Delta_\Psi^2 p}{\left| \frac{\partial \langle \Psi | \hat{p} | \Psi \rangle}{\partial \operatorname{Im} A_w} \right|^2} = \left(\frac{\hbar}{2\gamma} \right)^2 \left(\frac{1}{\Delta_\phi^2 p} \right) + \left(\frac{\hbar}{2\gamma} \right) \operatorname{Im} A_3 \left(\frac{p_3}{\left(\Delta_\phi^2 p \right)^2} \right), \Delta_\phi^2 p \neq 0$$

(15)

(this quantity is undefined when A_w is real valued). Inspection of eq.(15) reveals that for such measurements: (i) the sensitivity depends upon the variance control term $\operatorname{Im} A_w (p_3)$; (ii) the accuracy decreases (increases) as the measurement gets weaker (stronger) - i.e. as γ gets smaller (larger); and (iii) the accuracy can be arbitrarily increased (for a fixed $\gamma > 0$) via eq.(12) by choosing $\operatorname{Im} A_w (p_3) = - \left(\frac{\hbar}{2\gamma} \right) \Delta_\phi^2 p + \epsilon$, where ϵ is again a small positive real number.

In closing, it is important to note that the results discussed and developed above apply when the measurement interaction is instantaneous and the measurement is read from the pointer immediately after the interaction [14].

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V. ACKNOWLEDGEMENT

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